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Abstract

We reformulate the theory of the acceleration of charged particles in the recently proposed plasma wakefield acceleration scheme. Our treatment makes manifest the transient effects. To illustrate, we consider a point charge driving beam traversing a semi-infinite plasma. We show that it takes a few plasma wavelengths for transient effects to dissipate and give the wakefield solutions presently discussed in the literature. Thus, the transient effects place a limit on the length of the plasma.

The present generation of electron accelerators can provide accelerating E fields up to a few tens of megavolts per meter. Recently, a great deal of excitement has been generated by the plasma wakefield acceleration scheme.[1,2] In this technique, a bunched relativistic electron beam traverses a cold plasma, creating an electromagnetic field in its wake. This wakefield can generate huge accelerating gradients, up to hundreds and even thousands of Mev/m, which can act on a trailing electron bunch.

Present theoretical discussions of this scheme consider the long-time response of an infinite plasma when the $\mathbf{E}$ field is a function of $z - v_b t$, as is the driving bunch charge distribution, where z is the longitudinal coordinate defined by the direction of the velocity $\mathbf{v}_b$ of the driving beam.

S. van der Meer[3] recently stressed the importance of understanding the transient response of the plasma to the driving bunch to insure that the scheme does indeed yield the large gradients as is currently believed for the short time in which the trailing bunch is inside the plasma. Thus, we want to show to what extent the long-time analytic solutions for the wakefields are valid. Also, the plasma is localized in space. So, we want to be able to specify the boundary conditions on the surface surrounding the plasma. Moreover, at $t = 0$, we may have $\mathbf{E} \neq 0$ inside the plasma. This would be the case if a previous driving bunch had traversed the plasma. Thus, we may want to be able to vary the initial conditions inside the plasma volume.

Thus, our approach leads directly to a 3-dimensional general formalism which can be computer coded and which properly handles boundary and transient effects for an arbitrary charge distribution for the driving bunch.

To illustrate our approach, we consider a point charge distribution

$$\rho_b(\mathbf{r}) = -en_b(\mathbf{r}) = -\frac{e}{2\pi r}\delta(r)\delta(z - v_b t) \tag{1}$$

for the driving bunch incident on a semi-infinite cold plasma in the region $0 < z < \infty$, bounded by the surface $z = 0$. The linearized equations of motion are

$$\begin{aligned}
\frac{\partial n_1}{\partial t} + n_0(\nabla \circ \mathbf{v}_1) &= 0 && (2)\\
\frac{\partial \mathbf{v}_1}{\partial t} &= \frac{-e\mathbf{E}}{m} && (3)\\
\nabla \circ \mathbf{E} &= -4\pi e(n_1 + n_b) && (4)\\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} && (5)\\
\nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - \frac{4\pi e}{c}(n_b \mathbf{v}_b + n_0 \mathbf{v}_1), && (6)
\end{aligned}$$

where m is the mass of the electron, n_0 and n_1 are the background and perturbed plasma densities, $\mathbf{v}_1$ is the perturbed plasma velocity, and $\mathbf{E}$ is the electric field in the plasma. From the above equations we can derive

$$\frac{\partial^2 n_1}{\partial t^2} + \omega_p^2 n_1 = -\omega_p^2 n_b, \tag{7}$$

where

$$\omega_p = \left(\frac{4\pi e^2 n_0}{m}\right)^{\frac{1}{2}} \tag{8}$$

is the plasma frequency. The equation of motion for $\mathbf{E}$ can also be derived giving

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\omega_p^2}{c^2}\right)\mathbf{E}(\mathbf{r},t) = -4\pi e \nabla n_1 - 4\pi e \nabla n_b - \frac{4\pi e}{c^2}\frac{\partial(n_b \mathbf{v}_b)}{\partial t}. \tag{9}$$

Substituting n_b from Equation (1) into Equation (7) gives

$$n_1(\mathbf{r}) = -\frac{\omega_p}{v_b}\frac{\delta(r)}{2\pi r}u\left(t - \frac{z}{v_b}\right)\sin\omega_p\left(t - \frac{z}{v_b}\right), \tag{10}$$

where $u(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$ is the Heaviside unit step function. Thus, substituting for n_b and n_1 in Equation (9) we derive the following equation of motion for the longitudinal accelerating electric field:

$$\begin{aligned}\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - k_p^2\right)E_z(r,z,\theta,t) &= -2e\frac{\delta(r)}{r}\Big[(1-\beta_b^2)\delta'(z - v_b t) \\ &\quad + \left(\frac{\omega_p}{v_b}\right)^2 u\left(t - \frac{z}{v_b}\right)\cos\omega_p\left(t - \frac{z}{v_b}\right)\Big], \end{aligned} \tag{11}$$

where $k_p = \frac{\omega_p}{c}$ and $\beta_b = \frac{v_b}{c}$. At this point, other theoretical treatments look for solutions of the form $E_z = E_z(r,\xi)$, where $\xi = z - v_b t$. But this constraint leads to the long-time solution for the wakefield, after all transient effects have dissipated. We take a different approach. First, note that Equation (11) is the inhomogeneous Klein-Gordon equation.

Inside the plasma, namely $z > 0$, we have from Green's Theorem

$$\begin{aligned}
E_z(\mathbf{r},t) &= \int_0^{t^+} dt_0 \int_V dV_0 G(\mathbf{r},t \mid \mathbf{r}_0,t_0) q(\mathbf{r}_0,t_0) \\
&+ \frac{1}{4\pi}\int_0^{t^+} dt_0 \oint_S d\mathbf{S}_0 \circ (G\nabla_0 E_z - E_z \nabla_0 G) \\
&+ \frac{1}{4\pi c^2}\int_V dV_0 [G \frac{\partial E_z(\mathbf{r}_0,t_0)}{\partial t_0} - (\frac{\partial G}{\partial t_0}) E_z(\mathbf{r}_0,t_0)] \mid_{t_0=0}, \quad (12)
\end{aligned}$$

where $t^+ = t+\epsilon$ and $-4\pi q$ is the righthand side of Equation (11). We want to specify an arbitrary E_z on the boundary of the plasma at $z = 0$. From the method of images, the appropriate Green's function is

$$\begin{aligned}
G(\mathbf{r},t \mid \mathbf{r}_0,t_0) &= \frac{\delta(\tau - \frac{|\mathbf{r}-\mathbf{r}_0|}{c})}{\mid \mathbf{r}-\mathbf{r}_0 \mid} \\
&- \frac{k_p}{[\tau^2 - (\frac{R}{c})^2]^{\frac{1}{2}}} J_1(k_p c[\tau^2 - (\frac{R}{c})^2]^{\frac{1}{2}}) u(\tau - \frac{R}{c}) \\
&- \frac{\delta(\tau - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{\mid \mathbf{r}-\mathbf{r}' \mid} \\
&+ \frac{k_p}{[\tau^2 - (\frac{R'}{c})^2]^{\frac{1}{2}}} J_1(k_p c[\tau^2 - (\frac{R'}{c})^2]^{\frac{1}{2}}) u(\tau - \frac{R'}{c}), \quad (13)
\end{aligned}$$

where $R = \mid \mathbf{r}-\mathbf{r}_0 \mid$, $R' = \mid \mathbf{r}-\mathbf{r}' \mid$, $\tau = t - t_0$, and J_1 is the first order Bessel function of the first kind. To obtain a unique solution for the wakefield we choose the Dirichlet boundary condition $E_z = 0$ on the surface $z = 0$ and the Cauchy initial conditions $E_z = \frac{\partial E_z}{\partial t} = 0$ at $t = 0$ throughout the volume of the plasma.

By far the largest contribution to E_z comes from integrating the product of the second and first terms of Equations (11) and (13), respectively, giving

$$E_z^{(1)} = e\Big(\frac{\omega_p}{v_b}\Big)^2 \int_0^{\tilde{z}_0} dz_0 \frac{\cos\omega_p\big(t - \frac{[r^2+(z-z_0)^2]^{\frac{1}{2}}}{c} - \frac{z_0}{v_b}\big)}{[r^2+(z-z_0)^2]^{\frac{1}{2}}}, \tag{14}$$

where, for $\beta_b \simeq 1$, $\tilde{z}_0 = \frac{c^2t^2-r^2-z^2}{2(ct-z)}$. For small values of z, one should include the integral of the product of the second and third terms of Equations (11) and (13), respectively:

$$E_z^{(2)} = -e\Big(\frac{\omega_p}{v_b}\Big)^2 \int_0^{\approx{z}_0} dz_0 \frac{\cos\omega_p\big(t - \frac{[r^2+(z+z_0)^2]^{\frac{1}{2}}}{c} - \frac{z_0}{v_b}\big)}{[r^2+(z+z_0)^2]^{\frac{1}{2}}}, \tag{15}$$

where, for $\beta_b \simeq 1$, $\overset{\approx}{z}_0 = \frac{c^2t^2-r^2-z^2}{2(ct+z)}$ and $E_z = E_z^{(1)} + E_z^{(2)} +$ six other much smaller terms. Note that causality demands that $E_z = 0$ whenever r is so large that $\tilde{z}_0$ and $\overset{\approx}{z}_0 \leq 0$.

To make the connection to a real experiment, use our point charge model for the driving beam and consider the Argonne National Laboratory–University of Wisconsin, Madison plasma wakefield test facility presently being assembled.[4,5] A typical plasma density is $n_0 = 10^{13}/\mathrm{cm}^3$, giving $\lambda_p = 1.07$ cm and $\omega_p = 1.762 \times 10^{11}\ \mathrm{sec}^{-1}$. The number of particles in the driving bunch is typically $N_b = 10^{11}$ with $\beta_b = 0.9997$. As above, we model the plasma as having the one boundary surface $z = 0$.

If one looks for analytic solutions for the wakefield of the form

$E_z^{ANAL}(r,\xi)$, $\xi = z - v_b t$, one gets [6]

$$E_z^{ANAL}(r,\xi) = 2ek_p^2 K_0(k_p r) u(t - \frac{z}{c}) \cos \omega_p (t - \frac{z}{c}), \tag{16}$$

where K_0 is the zeroth order modified Bessel function of the second kind. In the derivation of this formula, transient and boundary effects are ignored.

In Figs. (1) and (2), we compare $E_z(r,z,t)$ from our calculation with the analytic solution $E_z^{ANAL}(r,\xi)$ versus time for fixed ξ. The value $\xi = 0.53$ cm corresponds to $\frac{1}{2}\lambda_p$ where $| E_z |$ is a maximum, and $\xi = 0.34$ cm just corresponds to an intermediate value of E_z. Since $t = 0$ corresponds to the time the driving bunch enters the plasma and since λ_p corresponds to 35.7 psec, we see that it takes about $1\frac{1}{2}$ and 2 plasma wavelengths in Figs. (1) and (2), respectively, for transient effects to dissipate so that the two solutions converge. Thus, for fixed ξ (as would be the case for a trailing witness beam pulse), there are substantial deviations of the analytic solutions from the exact solutions. It takes several λ_p for these effects to dissipate. As suggested by van der Meer,[3] one has to insure that the plasma is long enough that these transient effects dissipate over a sufficiently short fraction of the plasma length so that the enormous accelerating gradients indicated by Equation (16) are indeed achievable. In the Argonne–Wisconsin experiment, the length of the plasma is 10-30 times λ_p so that one can still expect the extremely large E_z over most of the plasma.

The above formalism allows one to take more complicated boundary conditions on the surface of the plasma, as well as more complicated initial conditions for E_z and $\frac{\partial E_z}{\partial t}$ at $t = 0$ throughout the volume of the plasma, such as for subsequent driving bunches. The application of this formalism to an arbitrary charge distribution for the driving bunch is straightforward.

References

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FIGURE CAPTIONS

Fig.1 Comparison of $E_z(r,z,t)$ (solid curve) from our formalism with $E_z^{ANAL}(r,\xi)$ (dashed curve) from Ref. (6) for $r=0.05$cm and $\xi=0.53$cm. The solutions converge in about $1.5\lambda_p$.

Fig. 2 Comparison of $E_z(r,z,t)$ (solid curve) from our formalism with $E_z^{ANAL}(r,\xi)$ (dashed curve) from Ref. (6) for $r=0.05$cm and $\xi=0.34$cm. The solutions converge in about $2\lambda_p$.

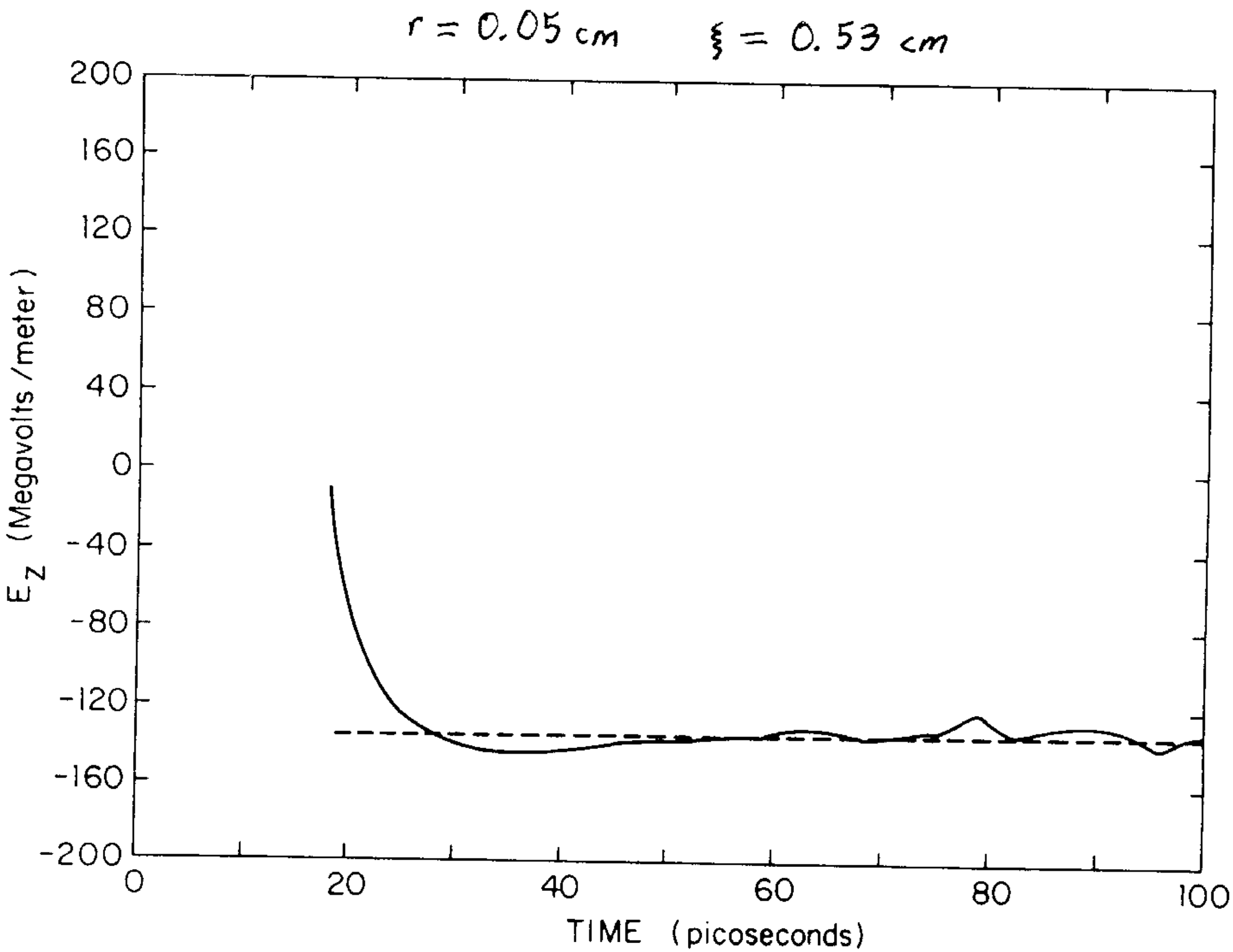
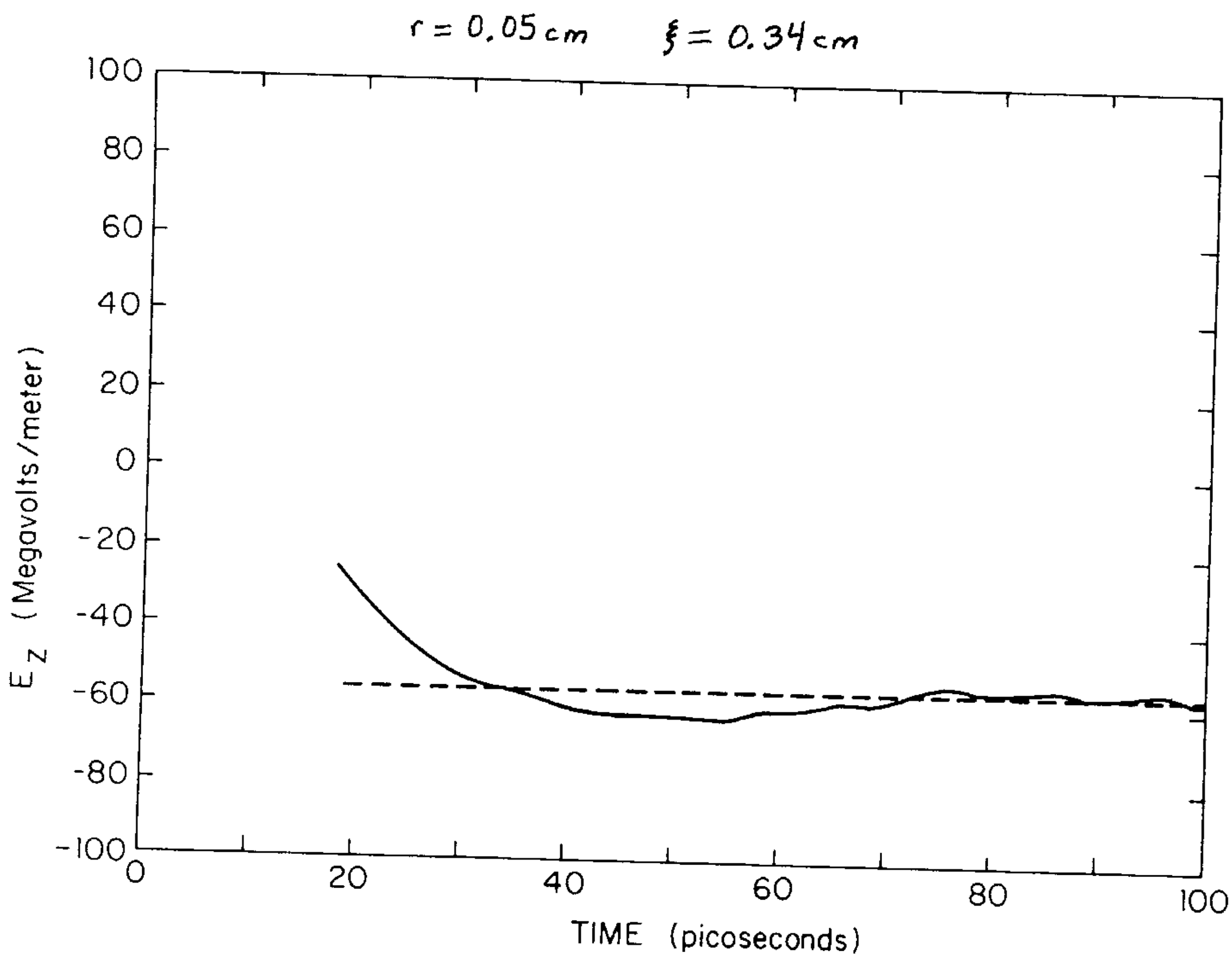

r = 0.05 cm
ξ = 0.53 cm
200
160
120
80
40
0
-40
-80
-120
-160
-200
E_Z (Megavolts /meter)
0
20
40
60
80
100
TIME (picoseconds)

FIG. 1



r = 0.05 cm   ξ = 0.34 cm
100
80
60
40
20
0
-20
-40
-60
-80
-100
E Z (Megavolts/meter)
0
20
40
60
80
100
TIME (picoseconds)

FIG. 2